Lecture No. 1: Coordinates, Graphs, Lines

Q 1: Solve the inequality $\frac{x-2}{x+1} > -1$.

Solution:

$$\frac{x-2}{x+1} > -1$$

$$\Rightarrow \frac{x-2}{x+1} + 1 > 0$$

$$\Rightarrow \frac{x-2+x+1}{x+1} > 0$$

$$\Rightarrow \frac{2x-1}{x+1} > 0$$

Now there are two possibilities. Either $2x-1>0$ and $x+1>0$ or $2x-1<0$ and $x+1<0$

Consider,

$2x-1>0$ and $x+1>0$

$$\Rightarrow x > \frac{1}{2} \text{ and } x > -1$$

$$\Rightarrow \left(\frac{1}{2}, +\infty\right) \cap (-1, +\infty)$$

Taking intersection of both intervals, we have

$$\left(\frac{1}{2}, +\infty\right) \text{...............(1)}$$

Similarly, if we consider,

$2x-1<0$ and $x+1<0$

$$\Rightarrow x < \frac{1}{2} \text{ and } x < -1$$

$$\Rightarrow \left(-\infty, \frac{1}{2}\right) \cap (-\infty, -1)$$

Taking intersection of both intervals, we have

$$(-\infty, -1) \text{.. (2)}$$

Combining (1) and (2), we have the required solution set. That is:

$$\left(\frac{1}{2}, +\infty\right) \cup (-\infty, -1)$$

Q 2: Solve the inequality and find the solution set of $3-\frac{1}{x}<\frac{1}{2}$.

Solution:

$3-\frac{1}{x}<\frac{1}{2}$

$\Rightarrow -\frac{1}{x}<\frac{1}{2}-3 \Rightarrow -\frac{1}{x}<-\frac{5}{2} \Rightarrow \frac{1}{x}>\frac{5}{2} \Rightarrow x<\frac{2}{5}$

So, solution set $=\left(-\infty,\frac{2}{5}\right)$

Q 3: List the elements in the following sets:

(i) $\{x : x^2+4x+4=0\}$ **(ii)** $\{x : x \text{ is an integer satisfying } -1<x<5\}$

Solution:

(i) Consider $x^2+4x+4=0$

$\Rightarrow x^2+2x+2x+4=0$

$\Rightarrow x(x+2)+2(x+2)=0$

$\Rightarrow (x+2)(x+2)=0$

$\Rightarrow (x+2)^2=0$

$\Rightarrow x+2=0$

$\Rightarrow x=-2$

Solution set: $\{-2\}$

(ii) Solution set: $\{0,1,2,3,4\}$

Q 4: Find the solution set for the inequality: $9+x>-2+3x$

Solution:

$9+x>-2+3x$

$\Rightarrow 9+2>3x-x \Rightarrow 11>2x \Rightarrow \frac{11}{2}>x$ or $x<\frac{11}{2}$

Hence,
Solution set: $\left(-\infty,\frac{11}{2}\right)$

Q 5: Solve the inequality $2<-1+3x<5$.

Solution:

$2<-1+3x<5$

$\Rightarrow 2+1<3x<5+1$

$\Rightarrow 3<3x<6$

$$\Rightarrow \frac{3}{3} < x < \frac{6}{3} \Rightarrow 1 < x < 2$$

Lecture No. 2: Absolute Value

Q 1: Solve for x, $\left|\frac{x+7}{4-x}\right| = 8.$

Solution:

$$\left|\frac{x+7}{4-x}\right| = 8$$

$\therefore \quad \frac{x+7}{4-x} = 8 \quad or \quad \frac{x+7}{4-x} = -8$

$\Rightarrow \quad x+7 = 8(4-x) \quad or \quad \Rightarrow \quad x+7 = -8(4-x)$

$\Rightarrow \quad x+7 = 32-8x \quad or \quad \Rightarrow \quad x+7 = -32+8x$

$\Rightarrow \quad x+8x = 32-7 \quad or \quad \Rightarrow \quad x-8x = -32-7$

$\Rightarrow \quad 9x = 25 \quad or \quad \Rightarrow \quad -7x = -39$

$\Rightarrow \quad x = \frac{25}{9} \quad or \quad \Rightarrow \quad x = \frac{39}{7}$

Q 2: Is the equality $\sqrt{b^4} = b^2$ valid for all values of b ? Justify your answer with appropriate reasoning.

Solution:

As we know that

$\sqrt{x^2} = x$ if x is positive or zero i.e $x \geq 0$,

$\therefore \quad \sqrt{b^4} = b^2$,

$\Rightarrow \quad \sqrt{(b^2)^2} = b^2$,

but b^2 is always positive , because if $b < 0$ then b^2 is always positive.

So the given equality always holds.

Q 3: Find the solution for: $|x^2 - 25| = x - 5$.

Solution:

$\because \quad |x^2 - 25| = x - 5$

$\Rightarrow x^2 - 25 = x - 5 \quad or \quad -(x^2 - 25) = x - 5,$

$\Rightarrow (x-5)(x+4) = 0 \quad or \quad (x+6)(-x+5) = 0,$

$\Rightarrow x = 5, -4 \quad or \quad x = -6,\ 5.$

For $x = -4$ in $|x^2 - 25| = x - 5,$

$\Rightarrow 9 = -9$ which is not possible.

For $x = -6$ in $|x^2 - 25| = x - 5,$

$\Rightarrow 11 = -11$ which is not possible.

$\therefore$ If $x = 5,$ then the given equation is clearly satisfied.

$\Rightarrow$ Solution is $x = 5.$

Q 4: Solve for x: $|6x - 8| - 10 = 8$.

Solution:

$\because \quad |6x - 8| - 10 = 8$

$\Rightarrow \quad |6x - 8| = 8 + 10 = 18$

$\Rightarrow \quad (6x - 8) = 18 \quad \text{or} \quad -(6x - 8) = 18$

$\Rightarrow \quad 6x = 26 \quad \text{or} \quad -6x = 10$

$\Rightarrow \quad x = \frac{13}{3} \quad \text{or} \quad x = -\frac{5}{3}$

$\therefore$ Solution is $x = -\frac{5}{3}, \frac{13}{3}.$

Q 5: Solve for x: $|x + 4| < 7$.

Solution:

Since $|x + 4| < 7$, so this inequality can also be written as
$-7 < x + 4 < 7$,
$-7 - 4 < x + 4 - 4 < 7 - 4$ (by subtracting 4 from the inequality),
$-11 < x < 3$,

So the solution set is $(-11, 3)$.

Lecture No. 3: Coordinate Planes and Graphs

Q 1: Find the x and y intercepts for $x^2 + 6x + 8 = y$

Solution:

x- Intercept can be obtained by putting $y = 0$ in the given equation i.e .,

$x^2 + 6x + 8 = 0$

its roots can be found by factorization.

$x^2 + 4x + 2x + 8 = 0$

$x(x+4) + 2(x+4) = 0$

$(x+2)(x+4) = 0$

either $x+2=0$ or $x+4=0$

this implies

$x = -2$ and $x = -4$

so, the x-intercepts will be $(-2,0)$ and $(-4,0)$

y-Intercept can be obtained by putting $x = 0$ in the given equation i.e.,

$$y = 8$$

So, the y-intercept will be (0,8).

Q 2: Find the x and y intercepts for $16x^2 + 49y^2 = 36$

Solution:

x- Intercept can be obtained by putting $y = 0$ in the given equation i.e.,

$$16x^2 + 0 = 36$$

$$x^2 = \frac{36}{16}$$

$$x = \pm\frac{6}{4} = \pm\frac{3}{2}$$

So, the x-intercept will be $\left(\frac{3}{2}, 0\right)$ and $\left(-\frac{3}{2}, 0\right)$.

y-Intercept can be obtained by putting $y = \pm\frac{6}{7}$

$$49y^2 + 0 = 36$$

$$y^2 = \frac{36}{49}$$

VU APEX CAMPUS	**vuapex.com.pk**	**vuapex.pk**
Contact Us:	**0322-8877744**	

$x = 0$ in the given equation i.e.,

So, the y-intercept will be $\left(0, \frac{6}{7}\right)$ and $\left(0, -\frac{6}{7}\right)$

Q 3: Check whether the graph of the function $y = x^4 - 2x^2 - 8$ is symmetric about x-axis and y-axis or not. (Do all necessary steps).

Solution:

Symmetric about x-axis:

If we replace $y \; to \; -y$, and the new equation will be equivalent to the original equation, the graph is symmetric about x-axis otherwise it is not.

Replacing $y \; to \; -y$, it becomes

$$-y = x^4 - 2x^2 - 8$$

Clearly, it is not equivalent to original equation, therefore, the graph is not symmetric about x-axis.

Symmetric about y-axis:

If we replace $x \; by \; -x$ and the new equation is equivalent to the original equation, the graph is symmetric about y-axis, otherwise it is not.

Replacing $x \; by \; -x$, it becomes

$$y = (-x)^4 - 2(-x)^2 - 8$$
$$= x^4 - 2x^2 - 8$$

Since the substitution made no difference to the equation, therefore, the graph will be symmetric about y-axis.

Q 4: Check whether the graph of the function $9x^2 + 4xy = 6$ is symmetric about x-axis and y-axis or not. (Do all necessary steps).

Solution:

Symmetric about x-axis:

If we replace $y \; to \; -y$, it becomes

$$9x^2 + 4x(-y) = 6$$
$$9x^2 - 4xy = 6$$

Clearly, it is not equivalent to original equation, therefore, the graph is not symmetric about x-axis.

Symmetric about y-axis:

Replacing $x \; by \; -x$, it becomes

$$9(-x)^2 + 4(-x)y = 6$$
$$9x^2 - 4xy = 6$$

Clearly, it is not equivalent to original equation, therefore, the graph is not symmetric about y-axis.

Symmetric about origin:

Replacing $x \; by \; -x$ and $y \; to \; -y$, it becomes

$$9(-x)^2 + 4(-x)(-y) = 6$$

$9x^2 + 4xy = 6$

Since the substitution made no difference to the equation, therefore, the graph will be symmetric about origin.

Q 5: Check whether the graph of the function $y=\frac{x^2-4}{x^2+1}$ is symmetric about x-axis and y-axis or not. (Do all necessary steps).

Solution:

Symmetric about y-axis:

Replacing $x\ by\ -x$, it becomes

$$y=\frac{(-x)^2-4}{(-x)^2+1}$$

$$=\frac{x^2-4}{x^2+1}$$

Since the substitution made no difference to the equation, therefore, the graph will be symmetric about y-axis.

Symmetric about origin:

Replacing $x\ by\ -x$ and $y\ to\ -y$, it becomes

$$-y=\frac{(-x)^2-4}{(-x)^2+1}$$

$$-y=\frac{x^2-4}{x^2+1}$$

Clearly, it is not equivalent to original equation, therefore, the graph is not symmetric about origin.

Lecture No. 4: Lines

Q 1: Find the slopes of the sides of the triangle with vertices (-1, 3), (5, 4) and (2, 8).
Solution: Let A(-1,3), B(5,4) and C(2,8) be the given points, then

Slope of side AB = $\frac{4-3}{5+1} = \frac{1}{6}$

Slope of side BC = $\frac{8-4}{2-5} = \frac{-4}{3}$

Slope of side AC = $\frac{3-8}{-1-2} = \frac{5}{3}$

Q 2: Find equation of the line passing through the point (1,2) and having slope 3.
Solution:
Point-slope form of the line passing through $P\left(x_1, y_1\right)$ and having slope m is given by the equation:

$y - y_1 = m\left(x - x_1\right)$

$\Rightarrow \quad y\text{-}2 = 3\left(x\text{-}1\right)$

$\Rightarrow \quad y - 2 = 3x - 3$

$\Rightarrow \quad y = 3x - 1$

Q 3: Find the slope-intercept form of the equation of the line that passes through the point (5,-3) and perpendicular to line $y = 2x + 1$.
Solution:

The slope-intercept form of the line with y-intercept b and slope m is given by the equation:
$y = mx + b$

The given line has slope 2, so the line to be determined will have slope $m = -\frac{1}{2}$.

Substituting this slope and the given point in the point-slope form: $y - y_1 = m\left(x - x_1\right)$, yields

$$y - (-3) = -\frac{1}{2}(x - 5)$$

$$\Rightarrow y + 3 = -\frac{1}{2}(x - 5)$$

$$\Rightarrow y = -\frac{1}{2}x + \frac{5}{2} - 3 \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}$$

Q 4: Find the slope and angle of inclination of the line joining the points (2, 3) and (-1, 2).

Solution:

If m is the slope of line joining the points (2, 3) and (-1, 2) then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{-1-2} = \frac{1}{3} \text{ is the slope}$$

Now angle of inclination is:

$$\tan\theta = m$$

$$\tan\theta = \frac{1}{3}$$

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.43^{o}$$

Q 5: By means of slopes, Show that the points lie on the same line

A (-3, 4); B (3, 2); C (6, 1)

Solution:

Slope of line through A(-3, 4) ; B(3, 2) $= \frac{2-4}{3+3} = -\frac{2}{6} = -\frac{1}{3}$

Slope of line through B(3, 2) ; C(6, 1) $= \frac{1-2}{6-3} = -\frac{1}{3}$

Slope of line through C(6, 1) ; A(-3, 4) $= \frac{4-1}{-3-6} = -\frac{3}{9} = -\frac{1}{3}$

Since all slopes are same, so the given points lie on the same line.

Lecture No. 5: Distance, Circles, Equations

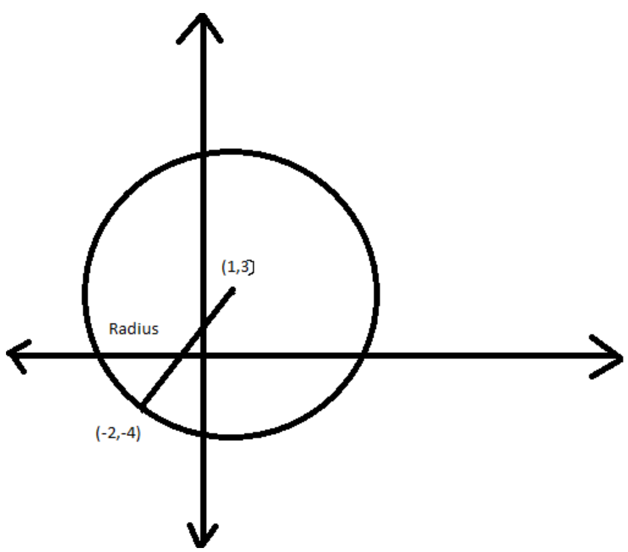
Q 1: Find the distance between the points (5,6) and (2,4) using the distance formula.
Solution:
The formula to find the distance between any two points (x_1, y_1) and (x_2, y_2) in the coordinate plane is given as

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The given points are $(5,6)$ and $(2,4)$, so the distance between these two points will be

$$\begin{aligned} \mathrm{d} &= \sqrt{(2-5)^2 + (4-6)^2} \\ &= \sqrt{(-3)^2 + (-2)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \end{aligned}$$

Q. 2: Find radius of the circle if the point (-2,-4) lies on the circle with center (1,3) .
Solution:
It is given that center of the circle is $(1,3)$. We are also given a point on the circle that is $(-2,-4)$ as shown below.



The radius of the circle will be the distance between the points (1,3) and (-2,-4). That is

$$\begin{aligned} \text{Radius} = \mathrm{d} &= \sqrt{[1-(-2)]^2 + [3-(-4)^2} \\ &= \sqrt{(3)^2 + (7)^2} \\ &= \sqrt{9+49} = \sqrt{58} \end{aligned}$$

Q 3: Find the coordinates of center and radius of the circle described by the following equation.

$$4x^2 + 4y^2 - 16x - 24y + 51 = 0$$

Solution:

The general form of the equation of circle is given as

$$4x^2 + 4y^2 - 16x - 24y + 51 = 0$$

It can be re-written as

$$(4x^2 - 16x) + (4y^2 - 24y) = -51$$

$$(2\,x)^2 - 2(8\,x) + (2\,y)^2 - 2(12\,y) = -51$$

In order to complete the squares on the left hand side, we have to add 16 and 36 on both sides, it will then become

$$(2\,x)^2 - 2(8\,x) + 16 + (2\,y)^2 - 2(12\,y) + 36 = -51 + 16 + 36$$

$$(2\,x)^2 - 2(2\,x)(4) + (4^2) + (2\,y)^2 - 2(2\,y)(6) + (6)^2 = 1$$

$$(2\,x - 4)^2 + (2\,y - 6)^2 = 1$$

$$(x - 2)^2 + (y - 3)^2 = \left(\frac{1}{4}\right)$$

Comparing it with the standard form of the equation, the center of the circle will be $(2, 3)$ and the radius will be $\frac{1}{2}$.

Q 4: Find the coordinates of center and radius of the circle described by the following equation.

$$2x^2 + 2y^2 + 6x - 8y + 12 = 0$$

Solution:

The general form of the equation of circle is given as

$$2x^2 + 2y^2 + 6x - 8y + 12 = 0$$

It can be re-written as

$$(2x^2 + 6x) + (2y^2 - 8y) = -12$$

$$(x^2 + 3x) + (y^2 - 4y) = -6$$

In order to complete the squares on the left hand side, we have to add $\frac{9}{4}$ and 4 on both sides, it will then become

$$(x^2 + 3x + \frac{9}{4}) + (y^2 - 4y + 4) = -6 + \frac{9}{4} + 4$$

$$(x^2 + 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 + (y)^2 - 2(y)(2) + (2)^2 = \frac{1}{4}$$

$$\left(x + \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{1}{4}$$

Comparing it with the standard form of the equation, the center of the circle will be $\left(-\frac{3}{2},2\right)$ and radius will be $\frac{1}{2}$.

Q 5: Find the coordinates of center and radius of the circle described by the following equation.

$$x^2 + y^2 - 4x - 6y + 8 = 0$$

Solution:

The general form of the equation of circle is given as

$$x^2 + y^2 - 4x - 6y + 8 = 0$$

This can be re-written as

$$(x^2 - 4x) + (y^2 - 6y) = -8$$

In order to complete the squares on the left hand side, we have to add 4 and 9 on both sides, it will then become

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) = -8 + 4 + 9$$

$$(\mathrm{x})^2 - 2(\mathrm{x})(2) + (2)^2 + (\mathrm{y})^2 - 2(\mathrm{y})(3) + (3)^2 = 5$$

$$(\mathrm{x}-2)^2 + (\mathrm{y}-3)^2 = 5$$

Comparing it with the standard form of the equation, the center of the circle will be (2,3) and the radius will be $\sqrt{5}$.

Q 6: Find the coordinates of the center and radius of the circle whose equation is

$$3x^2 + 6x + 3y2 + 18y - 6 = 0.$$

Solution:

$$\because \; 3x^2 + 6x + 3y^2 + 18y - 6 = 0,$$

$$\Rightarrow 3(x^2 + 2x + y^2 + 6y - 2) = 0, \quad (\because \text{ taking 3 as common})$$

$$\Rightarrow x^2 + 2x + y^2 + 6y - 2 = 0, \quad (\because \text{ dividing by 3 on both sides})$$

$$\Rightarrow x^2 + 2x + 1 + y^2 + 6y + 9 = 2 + 9 + 1,$$

$$\Rightarrow (x+1)^2 + (y+3)^2 = 12,$$

$$\Rightarrow (x+1)^2 + (y+3)^2 = (\sqrt{12})^2,$$

$$\Rightarrow (x-(-1))^2 + (y-(-3))^2 = (\sqrt{12})^2,$$

$\therefore$ Centre of the circle is (-1,-3) and radius is $\sqrt{12}$.

Q 7: Find the coordinates of the center and radius of the circle described by the following Equation

$$x^2 + y^2 - 6x - 8y = 0\,.$$

Solution:

$x^2 - 6x + y^2 - 8y = 0$, ($\because$ rearranging the term)

$x^2 - 6x + y^2 - 8y + (3)^2 = (3)^2$, $\left(\because \text{ adding } (3)^2 \text{ on both sides}\right)$

$(x^2 - 6x + 9) + y^2 - 8y = 9$,

$(x^2 - 6x + 9) + y^2 - 8y + (4)^2 = 9 + (4)^2$, $\left(\because \text{ adding } (4)^2 \text{ on both sides}\right)$

$(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$,

$(x-3)^2 + (y-4)^2 = 9 + 16$,

$(x-3)^2 + (y-4)^2 = (\sqrt{25})^2$, ______ eq.(1)

$\because$ $(x - x_0)^2 + (y - y_0)^2 = r^2$. ______ eq.(2)

The eq.(1) is now in the standard form of eq.(2). This equation represents a circle with the center at (3, 4) and with a radius equal to $\sqrt{25}$.

Q 8: Find the equation of circle with center $(3, -2)$ and radius 4.

Solution:

The standard form of equation of circle is

$(x - h)^2 + (y - k)^2 = r^2$,

Here $h = 3$, $k = -2$, $r = 4$,

$(x - 3)^2 + (y - (-2))^2 = 4^2$,

$x^2 - 6x + 9 + y^2 + 4 + 4y = 16$,

$x^2 + y^2 - 6x + 4y = 16 - 9 - 4$,

$x^2 + y^2 - 6x + 4y = 3.$

Q 9: Find the distance between A$(2, 4)$ and $B\,(8, 6)$ using the distance formula.

Solution:

The distance formula between two points (x_1, y_1) and (x_2, y_2) in a coordinate plane is given by

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,

$$
\begin{aligned}
d &= \sqrt{(8-2)^2+(6-4)^2}, \\
&= \sqrt{(6)^2+(2)^2}, \\
&= \sqrt{36+4}, \\
&= \sqrt{40}, \\
&= 2\sqrt{10}.
\end{aligned}
$$

Q 10: If the point $A(-1, -3)$lies on the circle with center B (3,-2), then find the radius of the circle.

Solution:

The radius is the distance between the center and any point on the circle, so find the distance:

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ ,$$

$$r = \sqrt{(3-(-1))^2 + (-2-(-3))^2} \ ,$$

$$= \sqrt{(3+1)^2 + (-2+3)^2} \ ,$$

$$= \sqrt{(4)^2 + (1)^2} \ ,$$

$$= \sqrt{16+1} \ ,$$

$$= \sqrt{17} \ ,$$

$$\approx 4.123.$$

Then the radius is $\sqrt{17}$, or about 4.123, rounded to three decimal places.

Lecture No. 6: Functions

Q 1: Find the natural domain and the range of the given function

$$h(x) = \cos^2(\sqrt{x}).$$

Solution:

As we know that the $\sqrt{x}$ is defined on non-negative real numbers $x \geq 0$. This means that the natural domain of $h(x)$ is the set of positive real numbers.

Therefore, the natural domain of $h(x) = [0, +\infty)$.

As we also know that the range of trigonometric function $\cos x$ is $[-1, 1]$.

The function $\cos^2\sqrt{x}$ always gives positive real values within the range 0 and 1 both inclusive.

From this we conclude that the range of $h(x) = [0, 1]$.

Q 2: Find the domain and range of function f defined by $f(x) = x^2 - 2$.

Solution:

$$\because \quad f(x) = x^2 - 2$$

The domain of this function is the set of all real numbers.

The range is the set of values that $f(x)$ takes as x varies. If x is a real number, x^2 is either positive or zero. Hence we can write the following:

$$x^2 \geq 0,$$

Subtract -2 on both sides to obtain

$$x^2 - 2 \geq -2.$$

The last inequality indicates that $x^2 - 2$ takes all values greater than or equal to -2. The range of function f is the set of all values of $f(x)$ in the interval $[-2, +\infty)$.

Q 3: Determine whether $y = \pm\sqrt{x+3}$ is a function or not? Justify your answer.

Solution:

$\because$ $y = \pm\sqrt{x+3}$, this is not a function because each value that is assigned to 'x' gives two values of y . So this is not a function. For example, if x=1 then

$$y = \pm\sqrt{1+3} \ ,$$
$$y = \pm\sqrt{4},$$
$$y = \pm 2.$$

Q 4: Determine whether $y=\frac{x+2}{x+3}$ is a function or not? Justify your answer.

Solution:

$$y=\frac{x+2}{x+3}$$

This is a function because each value that is assigned to '*x*' gives only one value of *y*
So this is a function. For example if *x*=1 then

$$y=\frac{1+2}{1+3},$$
$$y=\frac{3}{4},$$
$$y=0.75.$$

Q 5:

(a) Find the natural domain of the function $f(x)=\frac{x^2-16}{x-4}$.

(b) Find the domain of function f defined by $f(x)=\frac{-1}{(x+5)}$.

Solution:

(a)

$$f(x)=\frac{x^2-16}{x-4},$$
$$\Rightarrow f(x)=\frac{(x+4)(x-4)}{(x-4)},$$
$$=(x+4) \quad ; x\neq 4.$$

This function is defined at all real numbers x, except $x=4$.

(b)

$$f(x)=\frac{-1}{(x+5)}$$

This function consists of all real numbers x, except $x=-5$. Since $x=-5$ would make the denominator equal to zero and the division by zero is not allowed in mathematics.
Hence the domain in interval notation is given by $(-\infty,-5)\cup(-5,+\infty)$.

Lecture No. 7: Operations on Functions

Q 1: Consider the functions $f(x) = (x-2)^3$ and $g(x) = \frac{1}{x^2}$. Find the composite function $(fog)(x)$ and also find the domain of this composite function.

Solution:

Domain of $f(x) = -\infty < x < \infty = (-\infty, +\infty)$.

Domain of $g(x) = x < 0$ or $x > 0 = (-\infty, 0) \cup (0, +\infty)$.

$$
\begin{aligned}
fog(x) &= f(g(x)), \\
&= f\left(\frac{1}{x^2}\right), \\
&= \left(\frac{1}{x^2} - 2\right)^3.
\end{aligned}
$$

The domain fog consists of the numbers x in the domain of g such that $g(x)$ lies in the domain of f.

$\therefore$ Domain of $fog(x) = (-\infty, 0) \cup (0, +\infty)$.

Q 2: Let $f(x) = x + 1$ and $g(x) = x - 2$. Find $(f+g)(2)$.

Solution: From the definition,

$$
\begin{aligned}
(f+g)(x) &= f(x) + g(x), \\
&= x + 1 + x - 2, \\
&= 2x - 1.
\end{aligned}
$$

Hence, if we put $x = 2$, we get

$(f+g)(2) = 2(2) - 1 = 3.$

Q 3: Let $f(x) = x^2 + 5$ and $g(x) = 2\sqrt{x}$. Find $(gof)(x)$. Also find the domain of $(gof)(x)$.

Solution:

By definition,

$$
\begin{aligned}
(gof)(x) &= g(f(x)), \\
&= g(x^2 + 5), \\
&= 2\sqrt{x^2 + 5}.
\end{aligned}
$$

Domain of $f(x) = -\infty < x < \infty = (-\infty, +\infty)$.

Domain of $g(x) = x \geq 0 = [0, +\infty)$.

The domain of ***gof*** is the set of numbers ***x*** in the domain of ***f*** such that ***f*(*x*)** lies in the domain of ***g***.

Therefore, the domain of $g(f(x)) = (-\infty, +\infty)$.

Q 4: Given $f(x)=\dfrac{3}{x-2}$, and $g(x)=\sqrt{\dfrac{1}{x}}$. Find the domain of these functions. Also find the intersection of their domains.

Solution:

Here $f(x)=\dfrac{3}{x-2}$, so

domain of $f(x) = x<2$ or $x>2 = (-\infty,2)\cup(2,+\infty)$.

Now consider $g(x)=\sqrt{\dfrac{1}{x}}=\dfrac{1}{\sqrt{x}}$.

Domain of $g(x) = x>0 = (0,+\infty)$.

Also, intersection of domains:

domain of $f(x)\cap$ domain of $g(x) = (0,2)\cup(2,+\infty)$.

Q 5: Given $f(x)=\dfrac{1}{x^2}$ and $g(x)=\dfrac{2}{x-2}$, find $(f-g)(3)$.

Solution:

$$(f-g)(x) = f(x)-g(x),$$
$$=\frac{1}{x^2}-\frac{2}{x-2},$$
$$(f-g)(3)=\frac{1}{9}-\frac{2}{1}=\frac{1-18}{4}=\frac{-17}{9}.$$

Lecture No. 8-9
Lecture No.8: Graphs of Functions
Lecture No.9: Limits

Choose the correct option for the following questions:

1) If a vertical line intersects the graph of the equation $y = f(x)$ at two points, then which of the following is true?

- **I.** It represents a function.
- **II.** It represents a parabola.
- **III.** It represents a straight line.
- **IV.** It does not represent a function. Correct option

2) Which of the following is the reflection of the graph of $y = f(x)$ about *y*-axis?

- **I.** $y = -f(x)$
- **II.** $y = f(-x)$ Correct option
- **III.** $-y = -f(x)$
- **IV.** $-y = f(-x)$

3) Given the graph of a function $y = f(x)$ and a constant *c*, the graph of $y = f(x) + c$ can be obtained by________.

- **I.** Translating the graph of $y = f(x)$ up by c units. Correct option
- **II.** Translating the graph of $y = f(x)$ down by c units.
- **III.** Translating the graph of $y = f(x)$ right by c units.
- **IV.** Translating the graph of $y = f(x)$ left by c units.

4) Given the graph of a function $y = f(x)$ and a constant *c*, the graph of $y = f(x-c)$ can be obtained by________.

- **I.** Translating the graph of $y = f(x)$ up by c units.
- **II.** Translating the graph of $y = f(x)$ down by c units.

III. Translating the graph of $y = f(x)$ right by c units. Correct option

IV. Translating the graph of $y = f(x)$ left by c units.

5) Which of the following is the reflection of the graph of $y = f(x)$ about x-axis?

I. $y = -f(x)$ Correct option

II. $y = f(-x)$

III. $-y = -f(x)$

IV. $-y = f(-x)$

Q 6: If $\lim_{x\to 8^-} h(x) = 18 + c$ and $\lim_{x\to 8^+} h(x) = 7$ then find the value of '***c***' so that $\lim_{x\to 8} h(x)$ exists.

Solution:

For the existence of $\lim_{x\to 8} h(x)$ we must have $\lim_{x\to 8^-} h(x) = \lim_{x\to 8^+} h(x)$,

By placing the values we get

$18 + c = 7$,

$\Rightarrow c = 7 - 18 = -11.$

Q 7: Find the limit by using the definition of absolute value $\lim_{x\to 0^+} \frac{x}{|2x|}$.

Solution:

$\lim_{x\to 0^+} \frac{x}{|2x|}$,

where $|2x| = \begin{cases} 2x & x \geq 0, \\ -2x & x < 0. \end{cases}$

So $|2x| \to 2x$ as $x \to 0^+$.

$\therefore \lim_{x\to 0^+} \frac{x}{|2x|} = \lim_{x\to 0^+} \frac{x}{2x} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}.$

Q 8: Find the limit by using the definition of absolute value $\lim_{x\to 0^-} \frac{|x+5|}{x+5}$.

Solution:

$\lim_{x\to 0^-} \frac{|x+5|}{x+5}$

where $|x+5| = \begin{cases} x+5 & (x+5) \geq 0, \\ -(x+5) & (x+5) < 0. \end{cases}$

$\therefore \lim \frac{|x+5|}{} = \lim \frac{-(x+5)}{} = \lim(-1) = -1.$

$x \to 0^- \quad x+5 \quad x \to 0^- \quad x+5 \quad x \to 0^-$

Q 9: Evaluate: $\lim_{x \to \infty} \dfrac{x^2 - 3x + 1}{x^3 + 2x^2 - 5x + 3}$.

Solution:

$$\lim_{x \to \infty} \frac{x^2 - 3x + 1}{x^3 + 2x^2 - 5x + 3} = \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{3}{x^2} + \frac{1}{x^3}}{1 + \frac{2}{x} - \frac{5}{x^2} + \frac{3}{x^3}}, \quad (\because \text{ taking } x^3 \text{ as common})$$

$$= \frac{\frac{1}{\infty} - \frac{3}{\infty^2} + \frac{1}{\infty^3}}{1 + \frac{2}{\infty} - \frac{5}{\infty^2} + \frac{3}{\infty^3}}, \quad (\because \text{ on applying limit})$$

$$= \frac{0}{1}, \quad (\because \text{ any number divided by infinity is zero})$$

$$= 0. \quad \left(\because \frac{0}{1} = 0\right)$$

Q 10: Evaluate: $\lim_{z \to \infty} \dfrac{z^3 + 2z^2 - 5z + 3}{z^2 - 3z + 1}$.

Solution:

$$\lim_{z \to \infty} \frac{z^3 + 2z^2 - 5z + 3}{z^2 - 3z + 1} = \lim_{z \to \infty} \frac{1 + \frac{2}{z} - \frac{5}{z^2} + \frac{3}{z^3}}{\frac{1}{z} - \frac{3}{z^2} + \frac{1}{z^3}}, \quad (\because \text{ taking } x^3 \text{ as common})$$

$$= \frac{1 + \frac{2}{\infty} - \frac{5}{\infty^2} + \frac{3}{\infty^3}}{\frac{1}{\infty} - \frac{3}{\infty^2} + \frac{1}{\infty^3}}, \quad (\because \text{ on applying limit})$$

$$= \frac{1}{0},$$

$= \infty.$ $\left(\because \text{ any number divided by infinity is zero}\right)$ $\left(\because \frac{1}{0} = \infty\right)$

Lecture No. 10: Limits (Computational Techniques)

Q 1: Evaluate $\lim\limits_{x\to 5} \dfrac{x-5}{x^2-25}$.

Solution:

First we cancel out the zero in denominator by factorization:

$$\lim_{x\to 5} \frac{x-5}{x^2-25} = \lim_{x\to 5} \frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5} \frac{1}{x+5},$$

Now apply limit, we get:

$$\lim_{x\to 5} \frac{1}{x+5} = \frac{1}{10}$$

Q 2: Evaluate $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$.

Solution:
Factorize the numerator in the expression:

$$\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \lim_{x\to 2} \frac{x^2-5x-2x+10}{x-2}$$

$$= \lim_{x\to 2} \frac{x(x-5)-2(x-5)}{x-2}$$

$$= \lim_{x\to 2} \frac{(x-5)(x-2)}{x-2}$$

$$= \lim_{x\to 2}(x-5) = 2-5 = -3$$

Q 3: Evaluate $\lim\limits_{x\to 3} \dfrac{3x^3-9x^2+x-3}{x^2-9}$

Solution:

First we factorize the numerator and denominator and then apply its limit:

$$\lim_{x\to 3} \frac{3x^3-9x^2+x-3}{x^2-9} = \lim_{x\to 3} \quad 3x^2(x-3)+1(x-3) \quad = \lim_{x\to 3}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)(3x^2+1)(x-3)}{(x-3)(x+3)(3x^2+1)} \qquad (x+3)$$

$$= \frac{3(3)^2 + 1}{3+3} = \frac{28}{6} = \frac{14}{3}.$$

Q 4: Let $f(x) = \begin{cases} 3-x, & x<2 \\ \frac{x}{2}+1, & x>2 \end{cases}$

Determine whether $\lim\limits_{x\to 2} f(x)$ exist or not?

Solution:

For limit to exist, we must determine whether left-hand limit and right-hand limit at $x=2$ exist or not. So here we will find right hand and left hand limit.

Right-hand limit at $x=2$: $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}\left(\frac{x}{2}+1\right) = \frac{2}{2}+1 = 1+1 = 2$

Left-hand limit at $x=2$: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+}(3-x) = 3-2 = 1$

Clearly $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$, so limit does not exist.

Q 5: If $f(x) = \begin{cases} 3x+7, & 0<x<3 \\ 16, & x=3 \\ x^2+7, & 3<x<6 \end{cases}$, then show that $\lim\limits_{x\to 3} f(x) = f(3)$.

Solution:

Here $f(3)=16$. To find limit at $x=3$, we have to find the left-hand and right-hand limit at $x=3$, so: $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}\left(x^2+7\right) = 9+7 = 16$

And $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(3x+7) = 9+7 = 16$

Clearly $\lim\limits_{x\to 3^+} f(x) = 16 = \lim\limits_{x\to 3^-} f(x)$, so $\lim\limits_{x\to 3} f(x) = 16$

Lecture No. 11-12
Lecture 11: Limits (Rigorous Approach)
Lecture 12: Continuity

1) If $\lim_{x \to a} g(x) = L$ exists, then it means that for any $\varepsilon > 0$ $g(x)$ is in the interval_____.

- **I.** $(a - L, a + L)$
- **II.** $(a - \delta, a + \delta)$
- **III.** $(L - \delta, L + \delta)$
- **IV.** $(L - \varepsilon, L + \varepsilon)$ **Correct option is IV**

2) Using epsilon-delta definition, $\lim_{x \to 4} f(x) = 6$ can be written as ______.

- **I.** $|f(x) - 6| < \varepsilon$ whenever $0 < |x - 4| < \delta$ **Correct option is I**
- **II.** $|f(x) - 4| < \varepsilon$ whenever $0 < |x - 6| < \delta$
- **III.** $|x - 6| < \varepsilon$ whenever $0 < |f(x) - 4| < \delta$
- **IV.** $|f(x) - x| < \varepsilon$ whenever $0 < |6 - 4| < \delta$

3) Using epsilon-delta definition, our task is to find δ which will work for any_____.

- **I.** $\varepsilon < 0$
- **II.** $\varepsilon > 0$ **Correct option is II**
- **III.** $\varepsilon \geq 0$
- **IV.** $\varepsilon \leq 0$

4) Using epsilon-delta definition, $\lim_{x \to 1} f(x) = 2$ can be written as ______.

- **I.** $|x - 2| < \varepsilon$ whenever $0 < |f(x) - 1| < \delta$

II.I $|f(x) - x| < \varepsilon$ whenever $0 < |2 - 1| < \delta$

II. $|f(x) - 2| < \varepsilon$ whenever $0 < |x - 1| < \delta$ **Correct option is III**

IV. $|f(x) - 2| < \varepsilon$ whenever $0 < |x - 2| < \delta$

5) Which of the following must hold in the definition of limit of a function?

I. ε greater than zero

II. δ greater than zero

III. both ε and δ greater than zero **Correct option is III**

IV. none of these

Q 6: Show that $h(x)=2x^2-5x+3$ is a continuous function for all real numbers.

Solution:

To show that $h(x)=2x^2-5x+3$ is continuous for all real numbers, let's consider an arbitrary real number c. Now, we are to show that $\lim_{x\to c} f(x)= f(c)$

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c}(2x^2-5x+3)\\ &= 2c^2-5c+3\\ &= f(\text{c})\end{aligned}$$

Since, it is continuous on an arbitrary real number we can safely say that the given polynomial is continuous on all the real numbers.

Q 7: Discuss the continuity of the following function at $x=4$

$$f(x)=\begin{cases}-2x+8 & for\ x\le 4\\ \dfrac{1}{2}x-2 & for\ x>4\end{cases}.$$

Solution:

Given function is

$$f(x)=\begin{cases}-2x+8 & for\ x\le 4\\ \dfrac{1}{2}x-2 & for\ x>4\end{cases}$$

First of all, we will see if the function is defined at x=4. Clearly,

$$\begin{aligned}\text{f}(4)&=-2(4)+8\\ &=-8+8=0\end{aligned}$$

So, yes the function is defined at x =4.

Now, let's check the limit of the function at x =4

$$\begin{aligned}\lim_{x\to 4^-} f(\text{x}) &= \lim_{x\to 4^-}(-2\,\text{x}+8)\\ &=0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 4^+} f(\text{x}) &= \lim_{x\to 4^-}\left(\frac{1}{2}x-2\right)\\ &=0\end{aligned}$$

Since the left hand side limit and the right hand side limits exist and are equal so, the limit of the given function exist at x =4. Also,

$$\lim_{x \to 4} f(x) = f(4)$$

Hence, the function is continuous on the given point.

Q 8: Check the continuity of the following function at $x = 4$

$$g(x) = \begin{cases} x+4 & if \;\; x<1 \\ 2 & if \;\; 1 \le x < 4 \\ -5+x & if \;\; x \ge 4 \end{cases}$$

Solution:

Given function is

$$g(x) = \begin{cases} x+4 & if \;\; x<1 \\ 2 & if \;\; 1 \le x < 4 \\ -5+x & if \;\; x \ge 4 \end{cases}$$

First of all, we will see if the function is defined at x=4. Clearly,

$$\begin{aligned} g(4) &= -5+4 \\ &= -1 \end{aligned}$$

So the function is defined at x =4.

Now, let's check the limit of the function at x =4.

$$\begin{aligned} \lim_{x\to 4^-} g(x) &= \lim_{x\to 4^-}(2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 4^+} g(x) &= \lim_{x\to 4^+}(-5+x) \\ &= -1 \end{aligned}$$

Since the left hand side limit is not equal to the right hand side limit, therefore, the limit of the given function does not exist at x =4 and so the function is not continuous on the given point.

Q 9: Check the continuity of the function at $x = 3$**:** $f(\text{x}) = |x+3|$.

Solution:

The given function is

$$f(\text{x}) = |x+3|$$

Using the method of finding the limit of composite functions, we can write it as

$$\begin{aligned} \lim_{x\to 3} f(\text{x}) &= \lim_{x\to 3}|x+3| \\ &= \left|\lim_{x\to 3}\left(x+3\right)\right| \\ &= 6 \end{aligned}$$

Also,

$$f(3) = |3+3|$$
$$= |6| = 6$$

Since,

$$\lim_{x \to 3} f(x) = f(3)$$

Therefore, the given function is continuous at x=3.

Q 10: State why the following function fails to be continuous at *x=3*.

$$f(x) = \begin{cases} \dfrac{9-x^2}{3-x} & if \;\; x \neq 3 \\ 4 & if \;\; x = 3 \end{cases}$$

Solution:

The given function is

$$f(x) = \begin{cases} \dfrac{9-x^2}{3-x} & if \;\; x \neq 3 \\ 4 & if \;\; x = 3 \end{cases}$$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{9-x^2}{3-x}$$

$$= \lim_{x\to 3} \frac{(3-x)(3+x)}{3-x}$$

$$= \lim_{x\to 3}(3+x) = 6$$

$$f(3) = 4$$

Clearly,

$$\lim_{x\to 3} f(x) \neq f(3)$$

Therefore, the given function is not continuous at x=3.

Lecture No. 13: Limits and Continuity of Trigonometric Functions

Q 1: Determine whether $\lim_{x \to 0} \frac{1-\cos x}{|x|}$ exists or not?

Solution:
We shall find the limit as $x \to 0$ from the left and as $x \to 0$ from the right.
For left limit,

$$\lim_{x \to 0^-} \frac{1-\cos x}{|x|} = \lim_{x \to 0^-} \frac{1-\cos x}{-x} = -\lim_{x \to 0^-} \frac{1-\cos x}{-x} = 0 \qquad \because \textit{by corollary} \lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

For right limit,

$$\lim_{x \to 0^+} \frac{1-\cos x}{|x|} = \lim_{x \to 0^+} \frac{1-\cos x}{x} = 0 \qquad \because \textit{by corollary} \lim_{x \to 0} \frac{1-\cos x}{x} = 0 .$$

Since $\lim_{x \to 0^-} \frac{1-\cos x}{|x|} = 0 = \lim_{x \to 0^+} \frac{1-\cos x}{|x|}$, hence $\lim_{x \to 0} \frac{1-\cos x}{|x|}$ exist.

Q 2: Find the interval on which the given function is continuous:

$$y = \frac{x+3}{x^2 - 3x - 10}$$

Solution:

Given function is $y = \frac{x+3}{x^2 - 3x - 10}$

it is discontinuous only where denominator is '0' so

$x^2 - 3x - 10 = 0$

$x^2 - 5x + 2x - 10 = 0$

$x(x-5) + 2(x-5) = 0$

$(x-5)(x+2) = 0$

$x = 5, -2$

Points where the function is discontinuous *are* 5 and -2 so interval in which it is continous

$(-\infty, -2) \cup (-2, 5) \cup (5, +\infty)$

Q 3: Find the interval on which the given function is continuous:

$$y = \frac{1}{(x+2)^2} + 4$$

Solution:

Given function is $y = \dfrac{1}{(x+2)^2} + 4$

it is discontinuous only where denominator is '0' so

$(x+2)^2 = 0$

$x + 2 = 0$

$x = -2$

Point where the function is discontinuous is -2 so interval in which it is continous is

$(-\infty, -2) \cup (-2, +\infty)$

Q 4: Compute $\lim\limits_{x \to 0} \dfrac{\sin 3x}{4x}$.

Solution:

Here we will use the result that $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$.

$$\lim_{x \to 0} \frac{\sin 3x}{4x} = \frac{1}{4} \lim_{x \to 0} \frac{\sin 3x}{x} = \frac{1}{4} \lim_{x \to 0} \frac{\sin 3x}{x} \times \frac{3}{3} = \frac{3}{4} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{4}(1) = \frac{3}{4}$$

Q 5: Compute $\lim\limits_{\theta \to 0} \dfrac{\cos 2\theta + 1}{\cos\theta}$.

Solution: As we know $\cos 2\theta = 2\cos^2\theta - 1$, so

$$\lim_{\theta \to 0} \frac{\cos 2\theta + 1}{\cos\theta} = \lim_{\theta \to 0} \frac{2\cos^2\theta - 1 + 1}{\cos\theta} = \lim_{\theta \to 0} \frac{2\cos^2\theta}{\cos\theta} = \lim_{\theta \to 0} 2\cos\theta = 2\cos 0 = 2(1) = 2$$

Lecture No. 14: Rate of Change

Q 1: Find the instantaneous rate of change of $f(x) = x^2 + 1$ at x_0.

Solution:

Since $f(x) = x^2 + 1$ at x_0,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{\left((x_0+h)^2+1\right)-(x_0^2+1)}{h},$$

$$= \lim_{h\to 0}\frac{x_0^2+h^2+2x_0h+1-x_0^2-1}{h},$$

$$= \lim_{h\to 0}\frac{h^2+2x_0h}{h},$$

$$= \lim_{h\to 0}\frac{h(h+2x_0)}{h},$$

$$= \lim_{h\to 0}(h+2x_0),$$

$= 2x_0$ by applying limit, (Answer).

Q 2: Find the instantaneous rate of change of $f(x) = \sqrt{x+2}$ at an arbitrary point of the domain of f.

Solution:

Let a be any arbitrary point of the domain of f. The instantaneous rate of change of $f(x)$ at $x = a$ is

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\sqrt{x+2}-\sqrt{a+2}}{x-a},$$

$= \lim_{x\to a}\frac{\sqrt{x+2}-\sqrt{a+2}}{x-a}\times\frac{\sqrt{x+2}+\sqrt{a+2}}{\sqrt{x+2}+\sqrt{a+2}}$ by rationalizing,

$$= \lim_{x\to a}\frac{x+2-a-2}{(x-a)\left(\sqrt{x+2}+\sqrt{a+2}\right)},$$

$$= \lim_{x\to a}\frac{x-a}{(x-a)\left(\sqrt{x+2}+\sqrt{a+2}\right)},$$

$$= \lim_{x\to a}\frac{1}{\sqrt{x+2}+\sqrt{a+2}},$$

$= \frac{1}{\sqrt{a+2}+\sqrt{a+2}}$ by applying limit,

$= \frac{1}{2\sqrt{a+2}}$ (Answer).

Q 3: The distance traveled by an object at time t is $= f(t) = t^2$. Find the instantaneous velocity of the object at $t_0 = 4\ sec$.

Solution:

$$v_{inst} = m_{tan} = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0},$$

$$= \lim_{t_1 \to t_0} \frac{t_1^2 - 4^2}{t_1 - t_0},$$

$$= \lim_{t_1 \to t_0} \frac{t_1^2 - 16}{t_1 - t_0},$$

$$= \lim_{t_1 \to t_0} \frac{(t_1 + 4)(t_1 - 4)}{t_1 - t_0},$$

$$= \lim_{t_1 \to 4} \frac{(t_1 + 4)(t_1 - 4)}{(t_1 - 4)} \quad \text{because} \quad t_0 = 4\ sec,$$

$$= \lim_{t_1 \to 4} (t_1 + 4),$$

$= 4 + 4$ by applying limit,

$= 8$ (Answer).

Q 4: Find the instantaneous rate of change of $f(x) = x^3 + 1$ at $x_0 = 2$.

Solution:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h},$$

$$= \lim_{h \to 0} \frac{((2 + h)^3 + 1) - (2^3 + 1)}{h},$$

$$= \lim_{h \to 0} \frac{(2^3 + 3(2)^2 h + 3(2)h^2 + h^3 + 1) - (2^3 + 1)}{h},$$

$$= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 + 1 - (8 + 1)}{h},$$

$$= \lim_{h \to 0} \frac{9 + 12h + 6h^2 + h^3 - 9}{h},$$

$$= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h},$$

$$= \lim_{h \to 0} \frac{h(12 + 6h + h^2)}{h},$$

$$= \lim_{h \to 0} (12 + 6h + h^2),$$

$= 12$ (Answer).

Q 5:

(a) The distance traveled by an object at time t is $s = f(t) = t^2$. Find the average velocity of the object between $t = 2\ sec.$ and $t = 4\ sec.$

(b) Let $f(x) = \frac{1}{x-1}$. Find the average rate of change of f over the interval [5,7].

Solution:

(a) $\text{Avergae Velocity} = \frac{\underline{Distance\ travelled\ during\ interval}}{TIme\ Elapsed},$

$$v_{ave} = \frac{f(t_1)-f(t_0)}{t_1-t_0},$$

$$= \frac{f(4)-f(2)}{4-2},$$

$$= \frac{4^2-2^2}{2},$$

$$= \frac{16-4}{2},$$

$$= \frac{12}{2},$$

$$= 6 \text{ (Answer).}$$

(b) $\text{Avergae Velocity} = \frac{\underline{Distance\ travelled\ during\ interval}}{TIme\ Elapsed},$

$$m_{sec} = \frac{f(x_1)-f(x_0)}{x_1-x_0},$$

$$= \frac{f(7)-f(5)}{7-5},$$

$$= \frac{\frac{1}{7-1}-\frac{1}{5-1}}{2},$$

$$= \frac{\frac{1}{6}-\frac{1}{4}}{2},$$

$$= -\frac{1}{24}\ m/sec. \text{ (Answer).}$$

VU APEX CAMPUS	**vuapex.com.pk**	**vuapex.pk**
Contact Us:	**0322-8877744**	

Lecture No. 15: The Derivative

Q 1: Find the derivative of the following function by definition of derivative.

$$f(x) = 2x^2 - 16x + 35$$

Solution:

Given function is $f(x) = 2x^2 - 16x + 35$

By definition, the derivative of a function f(x) will be $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

For the given function, $f(x+h)$ will be as given below.

$$f(x+h) = 2(x+h)^2 - 16(x+h) + 35$$
$$= 2x^2 + 4hx + 2h^2 - 16x - 16h + 35$$

And so, the derivative will be

$$f'(x) = \lim_{h\to 0} \frac{2x^2 + 4hx + 2h^2 - 16x - 16h + 35 - (2x^2 - 16x + 35)}{h}$$
$$= \lim_{h\to 0} \frac{4hx + 2h^2 - 16h}{h}$$
$$= \lim_{h\to 0} \frac{h(4x + 2h - 16)}{h}$$
$$= \lim_{h\to 0}(4x + 2h - 16) = 4x - 16$$

Which is the required derivative of the given function.

Q 2: Find the derivative of the following function by definition of derivative.

$$f(x) = \frac{2}{5} + \frac{1}{2}x$$

Solution:

Given function is

$$f(x) = \frac{2}{5} + \frac{1}{2}x$$

By definition, the derivative of a function f(x) will be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

For the given function, $f(x+h)$ will be as given below.

$$f(x+h) = \frac{2}{5} + \frac{1}{2}(x+h)$$

And so, the derivative will be

$$f'(x) = \lim \frac{\frac{2}{5} + \frac{1}{2}(x+h) - \left(\frac{2}{5} + \frac{1}{2}x\right)}{}$$

$h \to 0 \qquad h$

$$= \lim_{h\to 0} \frac{h}{2h} = \frac{1}{2}$$

Which is the required derivative of the given function.

Q 3: Find the derivative of the following function by definition of derivative

$$g(t)=\frac{t}{t+1}$$

Solution:

Given function is $g(t)=\frac{t}{t+1}$

By definition, the derivative of a function g(t) will be

$$g'(\mathrm{t})=\lim_{h\to 0}\frac{\mathrm{g(t}+h)-g(\mathrm{t})}{h}$$

For the given function, $\mathrm{g(t}+h)$ will be as given below.

$$g(\mathrm{t}+\mathrm{h})=\frac{t+h}{t+h+1}$$

And so, the derivative will be

$$g'(t)=\lim_{h\to 0}\frac{1}{h}\left[\frac{t+h}{t+h+1}-\frac{t}{t+1}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(\mathrm{t}+\mathrm{h})(\mathrm{t}+1)-\mathrm{t}(\mathrm{t}+\mathrm{h}+1)}{(\mathrm{t}+\mathrm{h}+1)(\mathrm{t}+1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{t^2+t+th+h-t^2-th-t}{(\mathrm{t}+\mathrm{h}+1)(\mathrm{t}+1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{h}{(t+h+1)(\mathrm{t}+1)}\right]$$

$$=\frac{1}{(\mathrm{t}+1)^2}$$

Q 4: Find the equation of tangent line to the following curve at $x=1$

$$f(x)=\frac{1}{2x^2-x}$$

Solution:

Given function is

$$f(x)=\frac{1}{2x^2-x}$$

By definition, the derivative of a function f(x) will be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Given that x =1, it becomes

$$f'(x) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

For the given function, $f(1+h)$ will be as given below.

$$f(1+\mathrm{h}) = \frac{1}{2(1+\mathrm{h})^2 - (1+\mathrm{h})}$$

And so, the derivative at x=1 will be

$$f'(1)=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{2(1+h)^2-(1+h)}-\frac{1}{2(1)^2-(1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{2(1+h^2+2h)-(1+h)}-1\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{2h^2+3h+1}-1\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{h(2h-3)}{2h^2+3h+1}\right]=-3$$

Since the derivative at a point represents the slope of the tangent line at that point. So, we have m = -3. Thus, the equation of the tangent line with slope -3 will be

$$y-y_0=m(x-x_0)$$
$$y-1=-3(x-1)$$
$$y=-3x+4$$

Which is the required equation of tangent line.

Q 5: Find the equation of tangent line to the following curve at $x=2$

$$f(x)=\frac{x+2}{1-x}$$

Solution:
Given function is

$$f(x)=\frac{x+2}{1-x}$$

By definition, the derivative of a function f(x) will be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Given that x =2, it becomes

$$f'(x)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

And so, the derivative at x=2 will be

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{4+h}{-1-h}+4\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-3h}{-1-h}\right]=3$$

Since the derivative at a point represents the slope of the tangent line at that point. So, we have m = 3. Thus, the equation of the tangent line with slope 3 will be

$$y - y_0 = m(x - x_0)$$
$$y + 4 = 3(x - 2)$$
$$y = 3x - 10$$

Which is the required equation of tangent line.

Lecture No. 16: Techniques of Differentiation

Q 1: Differentiate $g(t)=\dfrac{t^2+4}{2t}$.

Solution:

$$\because g(t)=\frac{t^2+4}{2t},$$

$$\therefore g'(t)=\frac{2t\dfrac{d}{dt}(t^2+4)-(t^2+4)\dfrac{d}{dt}(2t)}{(2t)^2}, \quad (\because \text{ quotient rule})$$

$$=\frac{2t(2t)-(t^2+4)(2)}{4t^2}$$

$$=\frac{4t^2-2t^2-8}{4t^2}$$

$$=\frac{2t^2-8}{4t^2}$$

$$=\frac{t^2-4}{2t^2}.$$

Q 2: Evaluate $\dfrac{d}{dx}\left((x+1)(1+\sqrt{x})\right)$ at $x=9$.

Solution:

$$\frac{d}{dx}\left((x+1)(1+\sqrt{x})\right)=(x+1)\frac{d}{dx}(1+\sqrt{x})+(1+\sqrt{x})\frac{d}{dx}(x+1), \quad (\because \text{ product rule})$$

$$=(x+1)\left(\frac{1}{2\sqrt{x}}\right)+(1+\sqrt{x})(1),$$

$$=\frac{(x+1)}{2\sqrt{x}}+(1+\sqrt{x}),$$

by substituting $x=9$, $=\dfrac{(9+1)}{2\sqrt{9}}+(1+\sqrt{9})=\dfrac{10}{6}+4=\dfrac{10+24}{6}=\dfrac{34}{6}=\dfrac{17}{3}.$

Q 3: Differentiate the following functions:

i. $h(x) = (2x+1)(x+\sqrt{x})$.

ii. $g(x) = x^{-3}(5x^{-4}+3)$.

iii. $f(x) = \dfrac{x^3+1}{4x^2+1}$.

Solution (i): $h(x) = (2x+1)(x+\sqrt{x})$.

$$\frac{d}{dx}(h(x)) = (2x+1)\frac{d}{dx}(x+\sqrt{x}) + (x+\sqrt{x})\frac{d}{dx}(2x+1), \quad (\because \text{ product rule})$$

$$= (2x+1)\left(1+\frac{1}{2\sqrt{x}}\right) + (x+\sqrt{x})(2),$$

$$= (2x+1)\left(\frac{2\sqrt{x}+1}{2\sqrt{x}}\right) + (2x+2\sqrt{x}),$$

$$= 2x+1+\sqrt{x}+\frac{1}{2\sqrt{x}}+2x+2\sqrt{x},$$

$$= 4x+3\sqrt{x}+\frac{1}{2\sqrt{x}}+1.$$

Solution (ii): $g(x) = x^{-3}(5x^{-4}+3)$.

$\because \quad g(x) = x^{-3}(5x^{-4}+3) = 5x^{-7}+3x^{-3},$

$$\therefore \frac{d}{dx}(g(x)) = 5\frac{d}{dx}(x^{-7}) + 3\frac{d}{dx}(x^{-3}),$$

$$= 5(-7x^{-8}) + 3(-3x^{-4}),$$

$$= -35x^{-8} - 9x^{-4}.$$

Solution (iii): $f(x) = \dfrac{x^3+1}{4x^2+1}$.

$\because \quad f(x) = \dfrac{x^3+1}{4x^2+1},$

$$\therefore \frac{d}{dx}(f(x)) = \frac{(4x^2+1)\frac{d}{dx}(x^3+1) - (x^3+1)\frac{d}{dx}(4x^2+1)}{(4x^2+1)^2}, \quad (\because \text{ quotient rule})$$

$$= \frac{(4x^2 +1)(3x^2) - (x^3 +1)(8x)}{(4x^2 +1)^2},$$

$$= \frac{12x^4 + 3x^2 - (8x^4 + 8x)}{(4x^2 +1)^2},$$

$$= \frac{4x^4 + 3x^2 - 8x}{(4x^2+1)^2}.$$

Lecture No. 17: Derivatives of Trigonometric Function

Q 1: Find $\frac{dy}{dx}$ if $y = x^3 \cot x - \frac{3}{x^3}$.

Solution:

Given $y = x^3 \cot x - \frac{3}{x^3}$,

$\frac{dy}{dx} = \cot x \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(\cot x) - \frac{d}{dx}\left(\frac{3}{x^3}\right)$

$= \cot x\,(3x^2) + x^3(-cosec^2 x) - 3\frac{d}{dx}\left(\frac{1}{x^3}\right)$

$= 3x^2 \cot x - x^3 cosec^2 x + \frac{9}{x^4}$ (Answer).

Q 2: Find $\frac{dy}{dx}$ if $y = x^4 \sin x$ at $x = \pi$.

Solution:

$\because \frac{d}{dx}(f.g) = f.\frac{d}{dx}(g) + g.\frac{d}{dx}(f)$,

$y = x^4 \sin x$ at $x = \pi$,

$\frac{d}{dx} = \sin x \frac{d}{dx}(x^4) + x^4 \frac{d}{dx}(\sin x)$,

$= \sin x\,(4x^3) + x^4(\cos x)$,

$= 4x^3 \sin x + x^4 \cos x$,

$= 4\pi^3 \sin \pi + \pi^4 \cos \pi$, at $x = \pi$,

$= 4\pi^3(0) + \pi^4(-1)$,

$= -\pi^4$ (Answer).

Q 3: Find $f'(t)$ if $f(t) = \frac{2-8t+t^2}{sint}$.

Solution:

$\because \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}(f(x)) - f(x).\frac{d}{dx}(g(x))}{[g(x)]^2}$,

$f(t) = \frac{2-8t+t^2}{sint}$,

$f'(t) = \frac{[(\sin t)(-8+2t)] - [(2-8t+t^2)(\cos t)]}{(\sin t)^2}$,

$= \frac{[(2t-8)(\sin t)] - [(t^2-8t+2)(\cos t)]}{\sin^2 t}$ (Answer).

Q 4: Find $f'(y)$ if $(y) = \frac{\sin y + 3\tan y}{y^3 - 2}$.

Solution:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}(f(x)) - f(x).\frac{d}{dx}(g(x))}{[g(x)]^2},$$

$$f(y) = \frac{\sin y + 3\tan y}{y^3 - 2}.$$

$$f'(y) = \frac{(y^3-2)(\cos y + 3\sec^2 y) - [(\sin y + 3\tan y) + (3y^2)]}{(y^3-2)^2},$$

$$= \frac{(y^3-2)(\cos y + 3\sec^2 y) - [(\sin y + 3\tan y) + (3y^2)]}{y^6 - 4y^3 + 4} \quad \text{(Answer).}$$

Q 5: (a) Find $\frac{dy}{dx}$ if $y = (5x^2 + 3x + 3)(\sin x)$.

(b) Find $f'(t)$ if $(t) = 5t \sin t$.

Solution:

(a) $\because \frac{d}{dx}(f.g) = f.\frac{d}{dx}(g) + g.\frac{d}{dx}(f),$

$y = (5x^2 + 3x + 3)(\sin x)$,

$\frac{d}{dx}[(5\,x^2 + 3x + 3)(\sin x)] = (5\,x^2 + 3x + 3)(\cos x) + \sin x\,(10x + 3)$ (Answer).

(b) $\because \frac{d}{dx}(f.g) = f.\frac{d}{dx}(g) + g.\frac{d}{dx}(f),$

$f(t) = 5t \sin t$,

$\frac{d}{dt}(5\,t \sin t) = 5\,t \cos t + (\sin t)(5),$

$= 5\,t \cos t + 5 \sin t$ (Answer).

VU APEX CAMPUS	**vuapex.com.pk**	**vuapex.pk**
Contact Us:	**0322-8877744**	

Lecture No. 18: The Chain Rule

Q 1: Differentiate $y = \sqrt{5x^3 - 3x^2 + x}$ with respect to "*x*" using the chain rule.

Solution:

Given function is $y = \sqrt{5x^3 - 3x^2 + x}$.

Let $u = 5x^3 - 3x^2 + x.$

Then $y = \sqrt{u}$.

Using chain rule, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$

Here,

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}},$$

$$\frac{du}{dx} = 15x^2 - 6x + 1.$$

Then,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}}(15x^2 - 6x + 1),$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{5x^3 - 3x^2 + x}}(15x^2 - 6x + 1.$$

Q 2: Differentiate $y = \tan\sqrt{x} + \cos\sqrt{x}$ with respect to "*x*" using the chain rule.

Solution:

Given function is $y = \tan\sqrt{x} + \cos\sqrt{x}.$

Let $u = \sqrt{x}.$

Then $y = \tan(u) + \cos(u).$

Using chain rule, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Here,

$$\frac{dy}{du} = \sec^2 u - \sin u,$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

VU APEX CAMPUS	vuapex.com.pk	vuapex.pk
Contact Us:	**0322-8877744**	

Then,

$$\frac{dy}{dx} = (\sec^2 u - \sin u).\frac{1}{2\sqrt{x}},$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\left(\sec^2\sqrt{x} - \sin\sqrt{x}\right).$$

Q 3: Differentiate $y = 3\sin^2 x^5 + 4\cos^2 x^5$ with respect to "x" using the chain rule.

Solution:

Given function is $y = 3\sin^2 x^5 + 4\cos^2 x^5$.

Let $u = x^5$.

Then $y = 3\sin^2 u + 4\cos^2 u$.

Using chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$.

Here,

$$\begin{aligned}\frac{dy}{du} &= 3\times 2\sin u\cos u + 4\times 2\cos u(-\sin u),\\ &= 6\sin u\cos u - 8\cos u\sin u,\\ &= -2\sin u\cos u,\end{aligned}$$

$$\frac{du}{dx} = 5x^4.$$

Then,

$$\frac{dy}{dx} = 5x^4(-2\cos u\sin u),$$

$$\therefore \frac{dy}{dx} = -10x^4(\cos x^5 \sin x^5).$$

Q 4: Find $\dfrac{dy}{dx}$ if $y = \sqrt{\sec 4x}$ using chain rule.

Solution:

Given function is $y = \sqrt{\sec 4x}$.

Let $u = \sec 4x$.

Then $y = \sqrt{u}$.

Using chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$.

Here,

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}},$$

$$\frac{du}{dx} = 4\sec 4x\tan 4x.$$

VU APEX CAMPUS	vuapex.com.pk	vuapex.pk
Contact Us:	**0322-8877744**	

Then,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}}(4\sec 4x \tan 4x),$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{\sec 4x}}(4\sec 4x \tan 4x),$$

$$= 2\sqrt{\sec 4x}\ \tan 4x.$$

Q 5: Find $\frac{dy}{dt}$ if $y=\tan t^{\frac{2}{3}}$ using chain rule.

Solution:

Given function is $y=\tan t^{\frac{2}{3}}$.

Let $u=t^{\frac{2}{3}}$.

Then $y=\tan u$.

Using chain rule, $\frac{dy}{dt}=\frac{dy}{du}.\frac{du}{dt}$.

Here,

$$\frac{dy}{du}=\sec^2 u,$$

$$\frac{du}{dt}=\frac{2}{3}t^{-\frac{1}{3}}.$$

Then,

$$\frac{dy}{dt}=\sec^2 u\left(\frac{2}{3}t^{-\frac{1}{3}}\right),$$

$$\therefore \frac{dy}{dt}=\frac{2}{3t^{\frac{1}{3}}}\sec^2 t^{\frac{2}{3}}.$$

VU APEX CAMPUS	**vuapex.com.pk**	**vuapex.pk**
Contact Us:	**0322-8877744**	

Lecture No. 19: Implicit Differentiation

Q 1: Use implicit differentiation to find $\frac{dy}{dx}$ *if* $2xy = x + y - y^2$.

Solution:

$$Here\ 2xy = x + y - y^2.$$

Differentiate both sides w.r.t x :

$$\frac{d}{dx}(2xy) = \frac{d}{dx}(x + y - y^2)$$

$$\Rightarrow 2(x\frac{dy}{dx} + y(1)) = 1 + \frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$\Rightarrow 2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$$

$$\Rightarrow \frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$$

$$\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$$

Q 2: Use implicit differentiation to find $\frac{dy}{dx}$ *if* $x^5 + 3y^4 - y^3 + x^3y = 4$.

Solution:

Here $x^5 + 3y^4 - y^3 + x^3 y = 4.$

Differentiate both sides w.r.t x :

$$\Rightarrow 5x^4 + 12y^3\frac{dy}{dx} - 3y^2\frac{dy}{dx} + (x^3\frac{dy}{dx} + y(3x^2)) = 0$$

$$\Rightarrow 12y^3\frac{dy}{dx} - 3y^2\frac{dy}{dx} + x^3\frac{dy}{dx} = -5x^4 - 3x^2 y$$

$$\Rightarrow \frac{dy}{dx}\left(12y^3 - 3y^2 + x^3\right) = -5x^4 - 3x^2 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-5x^4 - 3x^2 y}{12y^3 - 3y^2 + x^3}$$

Q 3: Use imp~~licit d~~ifferentiation to find

Solution:

Here $y^2 - 2x = 1 - 2y$

Differentiate both sides w.r.t x :

$$\Rightarrow 2y\frac{dy}{dx} - 2 = -2\frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx} + 2\frac{dy}{dx} = 2$$

$$\Rightarrow y\frac{dy}{dx} + \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx}\left(y+1\right) = 1$$

$\frac{dy}{dx}$ if $y^2 - 2x = 1 - 2y$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{y+1}$$

Q 4: Find $\frac{dy}{dx}$ if $x^2 + y^2 = 4$

Solution:

here $x^2 + y^2 = 4$

Differentiate both sides, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Q 5: If $x^q = y^p$ then find $\frac{dy}{dx}$ in terms of variable "x".

Solution:

Here $x^q = y^p$ ....eq.(1)

Differentiate both sides w.r.t x:

$$qx^{q-1} = py^{p-1}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{qx^{q-1}}{py^{p-1}} \quad \text{........eq .(2)}$$

From eq.(1), we have $y = x^{\frac{q}{p}}$, put this value in eq.(2) in place of y, we will have:

$$\frac{dy}{dx} = \frac{qx^{q-1}}{p\left(x^{\frac{q}{p}}\right)^{p-1}} = \frac{qx^{q-1}}{px^{q-\frac{q}{p}}} = \frac{q}{p}x^{q-1-\left(q-\frac{q}{p}\right)} = \frac{q}{p}x^{-1+\frac{q}{p}}$$

Hence,

$$\frac{dy}{dx} = \frac{q}{p}x^{\frac{q}{p}-1}$$

Lecture No. 20: Derivatives of Logarithmic and Exponential Functions

Q 1: Differentiate: $y = (5-x)^{\sqrt{x}}$.

Solution:

$$\because \quad y = (5-x)^{\sqrt{x}},$$

taking log on both sides ,

$$\Rightarrow \ln y = \sqrt{x}\,\ln(5-x) \qquad \left(\because \ \ln m^n = n\ln m\right),$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.\ln(5-x) + \frac{1}{(5-x)}(-1).\sqrt{x} \ ,$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\ln(5-x)}{2\sqrt{x}} - \frac{\sqrt{x}}{(5-x)}\right).y \ ,$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\ln(5-x)}{2\sqrt{x}} - \frac{\sqrt{x}}{(5-x)}\right).(5-x)^{\sqrt{x}} .$$

Q 2: Differentiate $y = (\cos x)^{8x}$ with respect to 'x'.

Solution:

$$\because \quad y = (\cos x)^{8x} ,$$

taking log on both sides,

$$\Rightarrow \ln y = (8x)\ln(\cos x) , \qquad \left(\because \ \ln m^n = n\ln m\right),$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 8.\ln(\cos x) + \frac{1}{(\cos x)}.(-\sin x).(8x), \qquad \left(\because \ \frac{d}{dx}(\ln x) = \frac{1}{x} \ ; \ \frac{d}{dx}(\cos x) = -\sin x\right),$$

$$\Rightarrow \frac{dy}{dx} = \left(8\ \ln(\cos x) - \frac{8x\sin x}{\cos x}\right).y ,$$

$$\Rightarrow \frac{dy}{dx} = \left(8\ \ln(\cos x) - \frac{8x\sin x}{\cos x}\right)(\cos x)^{8x} .$$

Q 3: Differentiate $y = x^{\sin 5x}$ with respect to 'x'.

Solution:

$\because \quad y = x^{\sin 5x},$

Taking log on both sides ,

$\Rightarrow \ln y = (\sin 5x)\ln(x) \,, \qquad \left(\because \ln m^n = n\ln m\right),$

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = 5(\cos 5x).\ln(x) + \dfrac{1}{x}.(\sin 5x) \,, \qquad \left(\because \dfrac{d}{dx}(\ln x) = \dfrac{1}{x} \,;\, \dfrac{d}{dx}(\sin x) = \cos x\right),$

$\Rightarrow \dfrac{dy}{dx} = \left(5(\cos 5x).\ln(x) + \dfrac{\sin 5x}{x}\right).y \,,$

$\Rightarrow \dfrac{dy}{dx} = \left(5(\cos 5x).\ln(x) + \dfrac{\sin 5x}{x}\right)\left(x^{\sin 5x}\right).$

Q 4: Differentiate $y = x\, e^{3x+4}$.

Solution:

$\because \quad y = x\, e^{3x+4} \,,$

$\Rightarrow \dfrac{dy}{dx} = e^{3x+4} + xe^{3x+4}\dfrac{d}{dx}(3x+4) \,,$

$\Rightarrow \dfrac{dy}{dx} = e^{3x+4} + 3xe^{3x+4} \,.$

Q 5: Find the derivative of the function $y = \ln(2 + x^5)$ with respect to 'x'.

Solution:

$y = \ln(2 + x^5) \,,$

now taking the derivative of the function on both sides ,

$\dfrac{dy}{dx} = \dfrac{d}{dx}\{\ln(2 + x^5)\} \,,$

$\dfrac{dy}{dx} = \dfrac{1}{}\dfrac{d}{dx}(2 + x^5) \,,$

dx

$\dfrac{dy}{} =$ $\qquad dy =$

$$dx \quad (2+x^5)\ dx$$

$$\frac{1}{(2+x^5)}(0+5x^4)\ ,$$

$$\frac{5x^4}{(2+x^5)}\ .$$

Lecture No. 21: Applications of Differentiation

Q 1: If $f(x) = x^2 - 6x + 10$ then find the intervals where the given function is concave up and concave down respectively.

Solution:

Given function is $f(x) = x^2 - 6x + 10$

$$f'(x) = 2x - 6$$

$$f''(x) = 2 > 0$$

Since the second derivative is greater than zero for all values of x, so the given function is concave up on the interval $(-\infty, \infty)$ and it is concave down nowhere.

Q 2: If $f(x) = x^3 + 3x^2$ then find the intervals where the given function is concave up and concave down respectively.

Solution: The given function is $f(x) = x^3 + 3x^2$

$$f'(x) = 3x^2 + 6x$$

$$f''(x) = 6x + 6$$

For concave up

$$f''(x) = 6x + 6 > 0$$

$$6x > -6$$

$$x > -1$$

So, the given function is concave up on $(-1, \infty)$

For concave down

$$f''(x) = 6x + 6 < 0$$

$$= 6x < -6$$

$$= x < -1$$

So, the given function is concave down on $(-\infty, -1)$.

Q 3: If $f'(x) = 1 + 4x$ then find the intervals on which the given function is increasing or decreasing respectively.

Solution: It is given that $f'(x) = 1 + 4x$. The function will be increasing on all the values of x where first derivative is greater than zero. That is

$$f'(x) = 1 + 4x > 0$$

$$4x > -1$$

$$x > -\frac{1}{4}$$

Thus, the given function is increasing on $(-\frac{1}{4}, \infty)$.

The function will be decreasing on all the values of x where the first derivative is less than zero. That is

$$f'(x) = 1 + 4x < 0$$

$$4x < -1$$

$$x < -\frac{1}{4}$$

Thus, the given function is decreasing on $(-\infty, -\frac{1}{4})$.

Q 4: If $f'(x) = 2\,t - 2$ then find the intervals on which the given function is increasing or decreasing respectively.

Solution:

It is given that $f'(\mathrm{t}) = 2t - 2$. The function will be increasing on all the points where the first derivative is greater than zero. That is

$$f'(\mathrm{t}) = 2t - 2 > 0$$
$$2t > 2$$
$$t > 1$$

Thus, the given function is increasing on $(1, \infty)$

The given function will be decreasing on all the points where the first derivative is less than zero. That is

$$f'(\mathrm{t}) = 2t - 2 < 0$$
$$2t < 2$$
$$t < 1$$

Thus, the given function is decreasing on $(-\infty, 1)$.

Q 5: Discuss the concavity of the function $f(x) = (4 - x)(x + 4)$ on any interval using second derivative test.

Solution:

The given function is $f(x) = (4 - x)(x + 4)$

$$\begin{aligned} f(x) &= (4 - x)(x + 4) \\ &= 4x + 16 - x^2 - 4x \\ &= 16 - x^2 \\ f'(x) &= -2x \\ f''(x) &= -2 < 0 \end{aligned}$$

Since the second derivative is less than zero for all the values of x therefore, the given function is concave down on $(-\infty, \infty)$ and it is not concave up anywhere.

Lecture No. 22: Relative Extrema

Q 1: Find the vertical asymptotes for the function $f(x) = \dfrac{x+4}{x^2-25}$.

Solution:

The vertical asymptotes occur at the points where $f(x) \to \pm\infty$ i.e $x^2 - 25 = 0$

$$x^2 - 25 = 0$$
$$\Rightarrow x = \pm 5$$

Thus vertical asymptotes at $x = \pm 5$

Q 2: Find the horizontal asymptotes for the function $f(x) = \dfrac{x+4}{x^2-25}$.

Solution:

Horizontal asymptote can be found by evaluate $\lim\limits_{x\to+\infty} f(\text{x})$

$$\lim_{x\to+\infty} f(\text{x}) = \lim_{x\to+\infty} \frac{x+4}{x^2-25}$$

Divide numerator and denominator by x^2,

$$\lim_{x\to+\infty} f(\text{x}) = \lim_{x\to+\infty} \frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{25}{x^2}} = \frac{0+0}{1-0} = 0$$

Hence horizontal asymptotes at $y = 0$

Q 3: If $f(x) = 2x^4 - 16x^2$, determine all relative extrema for the function using First derivative test.

Solution:

First we will find critical points by putting $f'(\text{x}) = 0$

$$\Rightarrow 8x^3 - 32x = 0$$
$$\Rightarrow 8x\left(x^2 - 4\right) = 0 \Rightarrow x = 0,\ x = \pm 2$$

Because $f'(x)$ changes from negative to positive around −2 and 2, f has a relative minimum at x = −2 and x = 2, . Also, $f'(x)$ changes from positive to negative around 0, and hence, f has a relative maximum at x = 0.

Q 4: Find the relative extrema of $f(x) = \sin x - \cos x \, on \, [0, 2\pi]$ using 2nd derivative test.

Solution: First we will find critical points by putting $f^{/}(x) = 0$,

$$\Rightarrow \cos x + \sin x = 0$$

$$\Rightarrow \cos x = -\sin x$$

$$\Rightarrow \frac{\sin x}{\cos x} = -1 \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \; x = \frac{7\pi}{4}$$

Because $f^{/}(x)$ changes from negative to positive around $x = \frac{7\pi}{4}$, f has a relative minimum at $x = \frac{7\pi}{4}$. Also, $f^{/}(x)$ changes from positive to negative around $x = \frac{3\pi}{4}$, and hence, f has a relative maximum at $x = \frac{3\pi}{4}$.

Answer. relative maximum at $x = \frac{3\pi}{4}$, relative minimum at $x = \frac{7\pi}{4}$

Q 5: Find the critical points of $f(x) = x^{4/3} - 4x^{1/3}$.

Solution:

For critical point put

$$f^{/}(x) = 0 \Rightarrow \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = 0$$

$$\Rightarrow \frac{4}{3}x^{1/3} - \frac{4}{3x^{2/3}} = 0$$

$$\Rightarrow \frac{4}{3}\left(\frac{x-1}{x^{2/3}}\right) = 0$$

$$\Rightarrow \frac{x-1}{x^{2/3}} = 0$$

critical points occur where numerator and denominator is zero.i.e

$$x - 1 = 0, \; x^{2/3} = 0$$

$$\Rightarrow x = 1, \quad x = 0$$